

Upwind DG Methods for Second Order Wave Equations

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Collaborators in DG work:

Energy DG: Daniel Appelö (Michigan State) and Lu Zhang (Columbia)

Galerkin Difference Methods: Jeff Banks (RPI), Brett Buckner (Mathworks), Fritz Juhnke (Bethel)

GD-DG on Hybrid Grids: Jeff Banks, Jeremy Kozdon (NPS), Lucas Wilcox (NPS)

Goals:

- i. For first order hyperbolic systems (with principal parts) satisfying energy estimates - Friedrichs systems - upwind DG methods are general and fairly well-understood. We would like to develop similar constructions for second order wave equations - for example the regular hyperbolic systems following from an action principle as defined by Christodoulou. The methods we construct will have a discrete energy estimate (perhaps only for the principal part):

$$\frac{d\mathcal{E}_h}{dt} = \text{Flux Terms} + \text{LowerOrderTerms}$$

and the upwind fluxes will enforce:

$$\text{Flux Terms} \leq 0.$$

If properly constructed this should lead to stability for the semidiscrete problem.

- ii. On simple meshes finite difference methods are more efficient than finite element methods on unstructured grids. We wish to develop finite difference methods which are also DG methods satisfying all the usual DG theory.

Current Status:

i. We have developed an energy-DG formulation and shown how to satisfy the desired energy estimates for:

- Linear and nonlinear systems whose Lagrangian is of the form (Kinetic Energy) – (Potential Energy) - see Appelö and TH (SINUM 2015, CMAME 2018), Zhang, Appelö and TH (JCP 2020 and preprints),
- Scalar advective wave equations - see Zhang, Appelö and TH (SINUM 2019).

To solve the full problem (including some formulations of the Einstein equations) we need to combine the two developments - the current issue is the precise comparison of the two types of fluxes which appear.

ii. We have developed a theory of Galerkin difference methods and shown how it works with simpler interior penalty DG methods for second order equations - see Banks, TH and others (JCP 2016, JCP 2018, JCP 2019, JSC 2019, SISC 2019, SISC 2020).

The Energy-DG idea: evolve a variable v **weakly** equal to an advective time derivative and introduce an upwind splitting of the energy flux.

Consider the case where for an element Ω with $\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t}$:

$$\mathcal{E} = \int_{\Omega} \frac{1}{2} |\mathbf{v}|^2 + G(\mathbf{u}, D\mathbf{u})$$

and

$$\frac{d\mathcal{E}}{dt} = \int_{\partial\Omega} \sum_{j,k} v_j \frac{\partial G}{\partial u_{j,k}} n_k$$

we can define an upwind energy flux based on a flux splitting (more complicated constructions may be needed for the most general case):

$$\sum_k v_j \frac{\partial G}{\partial u_{j,k}} n_k = \frac{1}{4} \left(v_j + \sum_k \frac{\partial G}{\partial u_{j,k}} n_k \right)^2 - \frac{1}{4} \left(v_j - \sum_k \frac{\partial G}{\partial u_{j,k}} n_k \right)^2$$

and choose desired flux terms by insisting (where now the normal refers to each element)

$$v_j^* - w_j^* = v_j - \sum_k \frac{\partial G}{\partial u_{j,k}} n_k$$

The weak form which we then discretize is:

$$\begin{aligned} \int_{\Omega} \left(\sum_k \tilde{G}_{j,k}(\phi_u, \mathbf{u}, D\phi_u, D\mathbf{u}) \frac{\partial}{\partial x_k} + \hat{G}_j(\phi_u, \mathbf{u}, D\phi_u, D\mathbf{u}) \right) \left(\frac{\partial u_j}{\partial t} - v_j \right) &= \int_{\partial\Omega} \sum_k \tilde{G}_{j,k} n_k (v_j^* - v_j) \\ \int_{\Omega} \phi_{v,j} \left(\frac{\partial v_j}{\partial t} + \frac{\partial G}{\partial u_j}(\mathbf{u}, D\mathbf{u}) \right) + \sum_k \frac{\partial \phi_{v,j}}{\partial x_k} \frac{\partial G}{\partial u_{j,k}}(\mathbf{u}, D\mathbf{u}) &= \int_{\partial\Omega} \phi_{v,j} w_j^*. \end{aligned}$$

Here \tilde{G} and \hat{G} must depend linearly on the test functions ϕ_u and reduce to $\frac{\partial G}{\partial u_{j,k}}$ and $\frac{\partial G}{\partial u_j}$ if $\phi_u = \mathbf{u}$.

Then we can prove energy stability $\frac{d\mathcal{E}}{dt} \leq 0$ and 1/2-order suboptimal convergence - we generally observe optimal convergence.

A simpler alternative which one can also use at least in these simpler cases can be based on symmetric interior penalty formulations (SIPDG):

$$\int_{\Omega} \phi_{u,j} \left(\frac{\partial^2 u_j}{\partial t^2} + \frac{\partial G}{\partial u_j}(\mathbf{u}, D\mathbf{u}) \right) + \sum_k \frac{\partial \phi_{u,j}}{\partial x_k} \frac{\partial G}{\partial u_{j,k}}(\mathbf{u}, D\mathbf{u}) = \int_{\partial\Omega} \phi_{u,j} w_j^* + \text{penalty terms}.$$

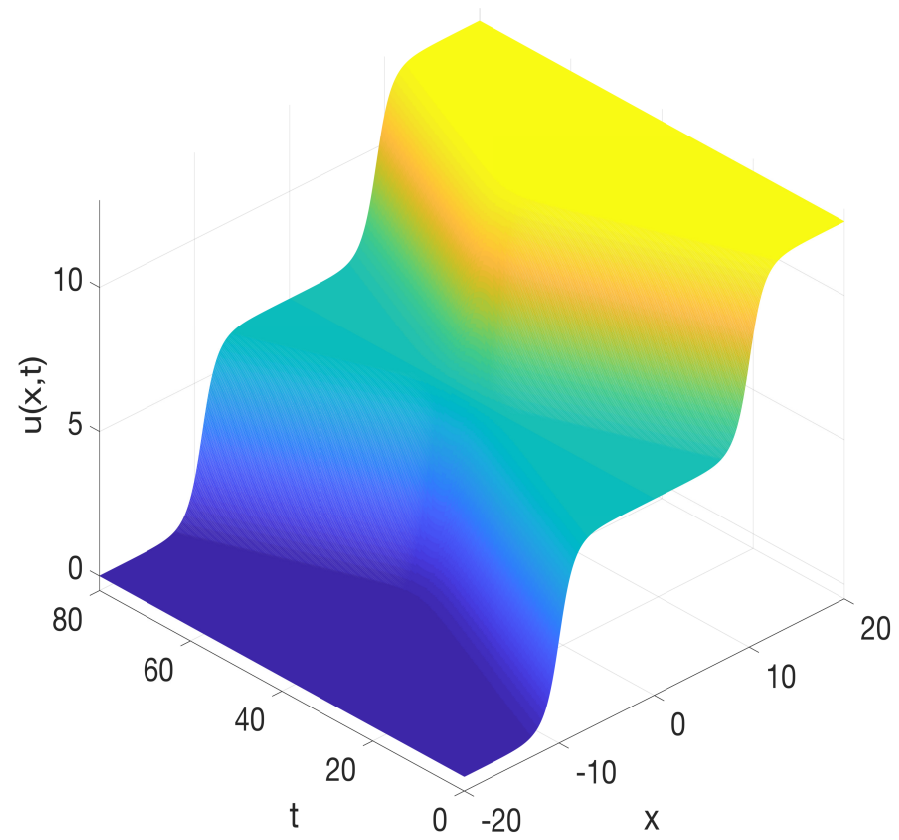
The disadvantage of the upwind SIPDG method is that stability requires mesh-depedent penalty paramters. Stability for energy-DG does not. The advantage is that the local stiffness matrix does not need to be inverted. For this reason it is easier to use with the Galerkin difference methods which employ elements the size of grid blocks.

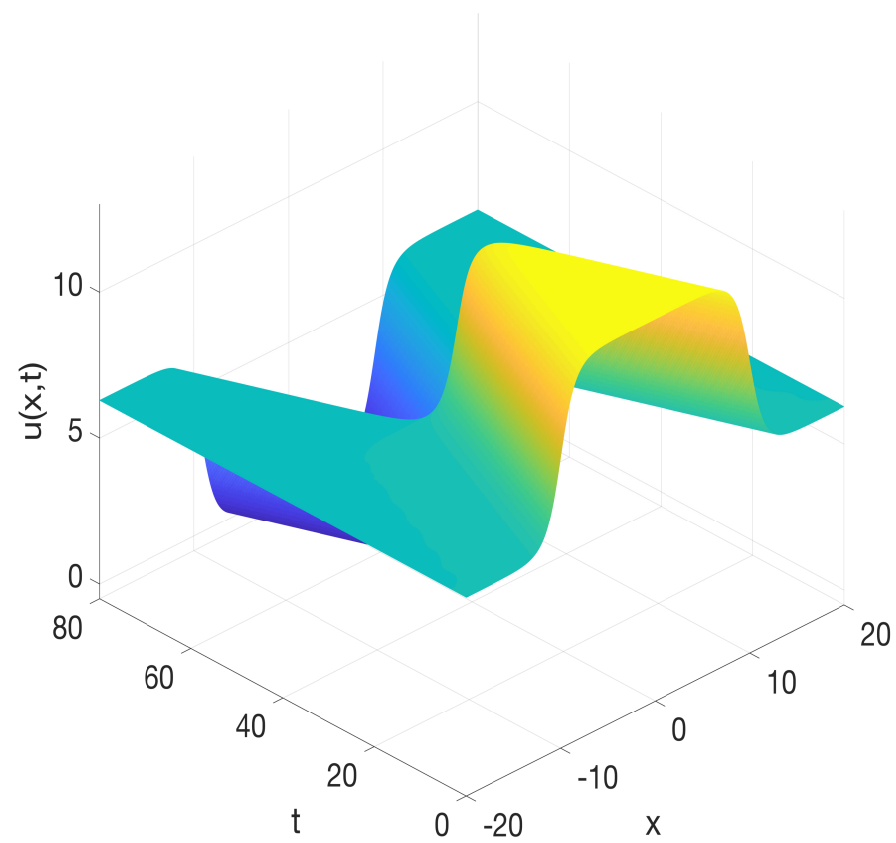
Too many equations so lets take a break for pictures!

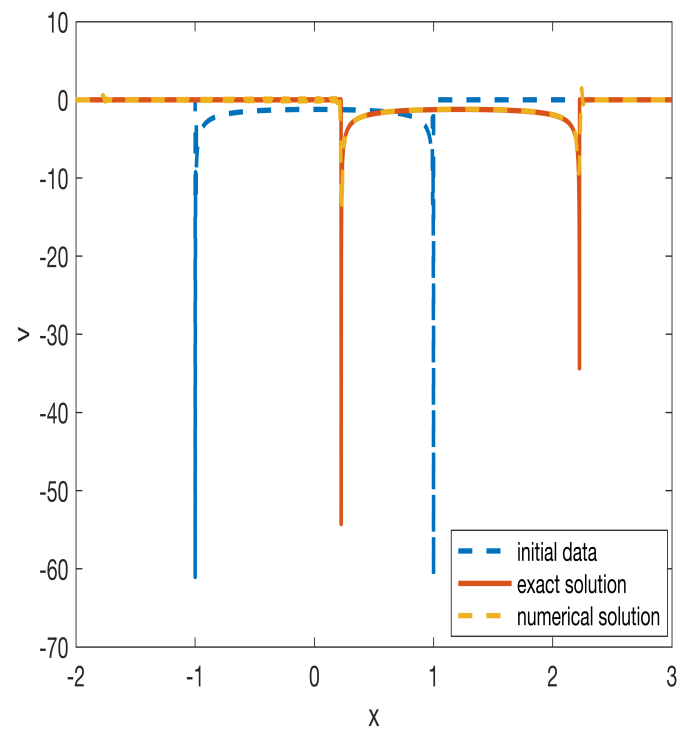
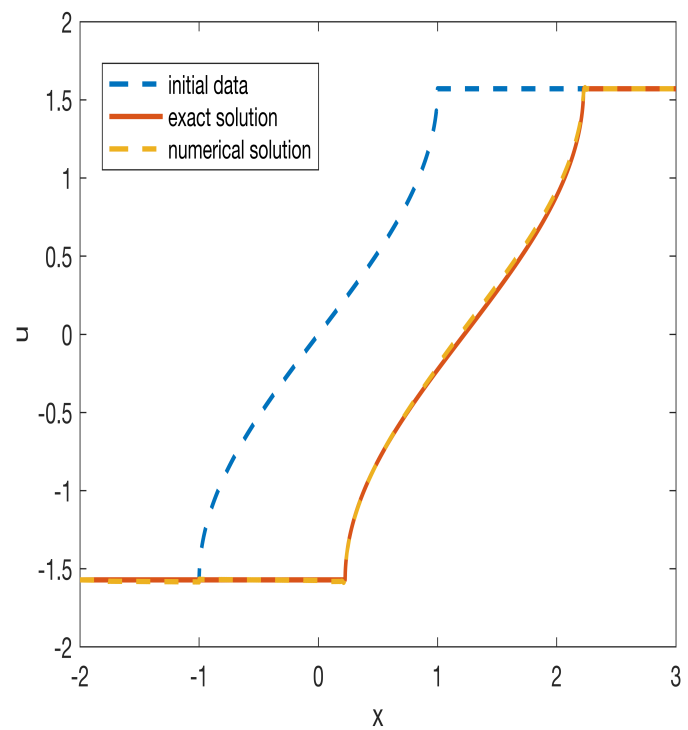
Two scalar nonlinear examples:

Sine-Gordon $G = \frac{c^2}{2}|\nabla u|^2 + 1 - \cos u$ - we show results from a fifth order method applied to a collision of kink-kink and kink-antikink solitons

Nematic $G = \frac{\alpha \cos^2 u + \beta \sin^2 u}{2}|\nabla u|^2$. This is interesting as the solution develops a singularity - u is continuous but v blows up - however weak solutions are not unique and it is not which one is correct. We evolve a non-smooth exact traveling wave solution.







Generalization to the wave equation with advection:

$$\mathcal{E} = \int_{\Omega} \frac{1}{2} \left(\frac{\partial u}{\partial t} + \mathbf{w} \cdot \nabla u \right)^2 + \frac{c^2}{2} |\nabla u|^2.$$

Set

$$v = \frac{\partial u}{\partial t} + \mathbf{w} \cdot \nabla u.$$

and solve

$$\begin{aligned} \int_{\Omega} c^2 \nabla \phi_u \cdot \nabla \left(\frac{\partial u}{\partial t} + \mathbf{w} \cdot \nabla u - v \right) &= \int_{\partial\Omega} c^2 (v^* - v) \nabla \phi_u \cdot \mathbf{n} \\ &\quad - c^2 \nabla \phi_u \cdot (\nabla u^* - \nabla u) \mathbf{w} \cdot \mathbf{n}, \\ \int_{\Omega} \phi_v \frac{\partial v}{\partial t} + \phi_v \mathbf{w} \cdot \nabla v + c^2 \nabla u \cdot \nabla \phi_v &= \int_{\Omega} \phi_v f \\ &\quad + \int_{\partial\Omega_j} c^2 \phi_v \nabla u^* \cdot \mathbf{n} - (v^* - v) \phi_v \mathbf{w} \cdot \mathbf{n}. \end{aligned}$$

Choosing upwind fluxes as above we can prove the same results in the subsonic case $|w \cdot n| < c$ and also with fully upwind fluxes in the supersonic case: $|w \cdot n| > c$.

Why consider finite difference methods? Efficiency - besides the advantages of more regular data structures finite difference methods at high orders do not suffer from the artificial stiffness of polynomial-based finite element methods.

This unfavorable scaling of the DG derivative matrix is a fundamental property of polynomials - the Markov/Bernstein inequalities. For polynomials of degree m on the standard interval, $[-1, 1]$ we have:

$$\left\| \frac{dq}{dx}(x) \right\| \leq \min \left(m^2, \frac{m}{\sqrt{1-x^2}} \right) \|q\|_{\infty}$$

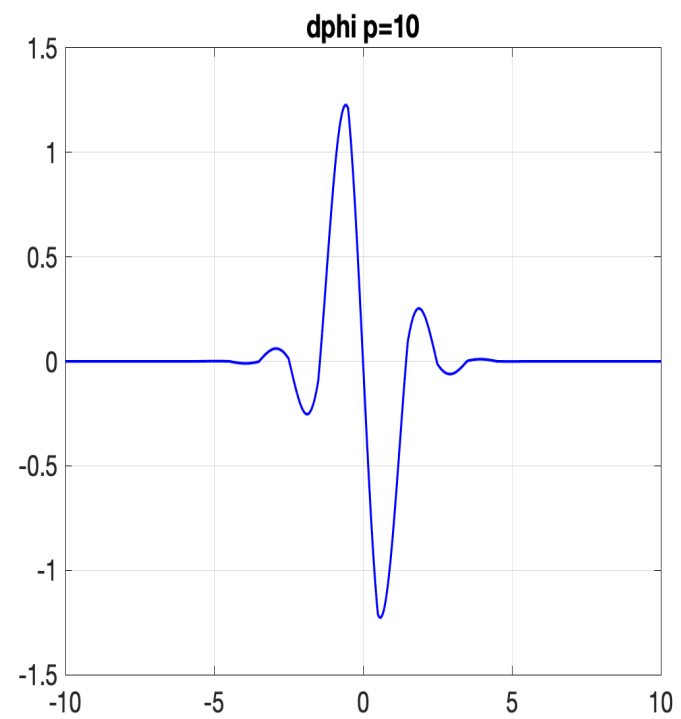
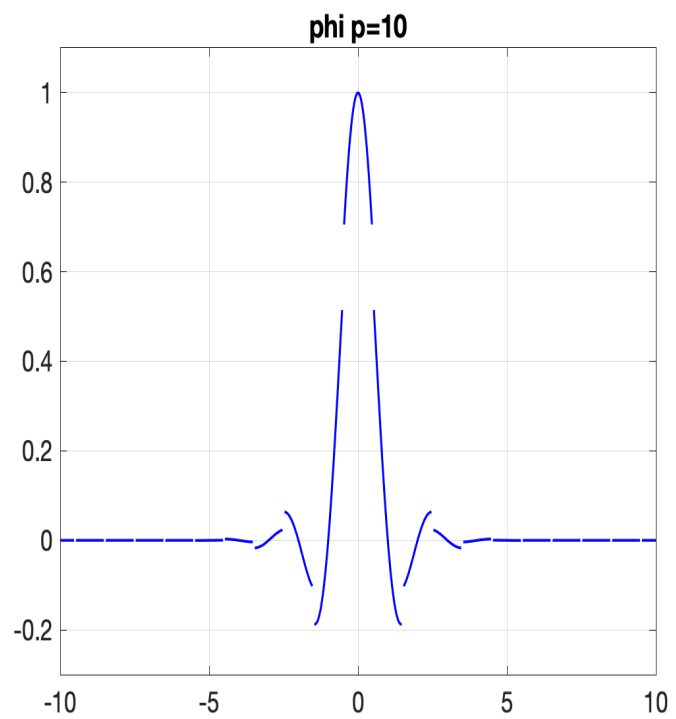
These show that polynomials can have boundary layers at element edges.

Linear m -dependence for the derivative requires the use of data outside the interval where the derivative is to be computed - as in difference formulas.



Discontinuous Galerkin Difference methods - here the idea is simply to define the basis on a (possibly mapped) uniform grid for which a trial/test function restricted to a cell is the degree- p tensor-product interpolant based on cell-centered data. (Continuous Galerkin Difference methods use data on cell boundaries.) We plot the basis functions below.

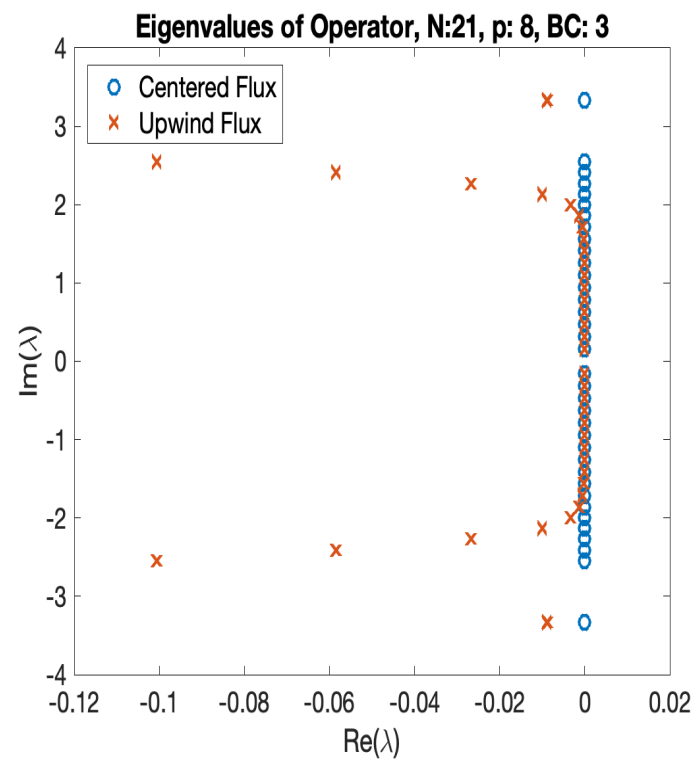
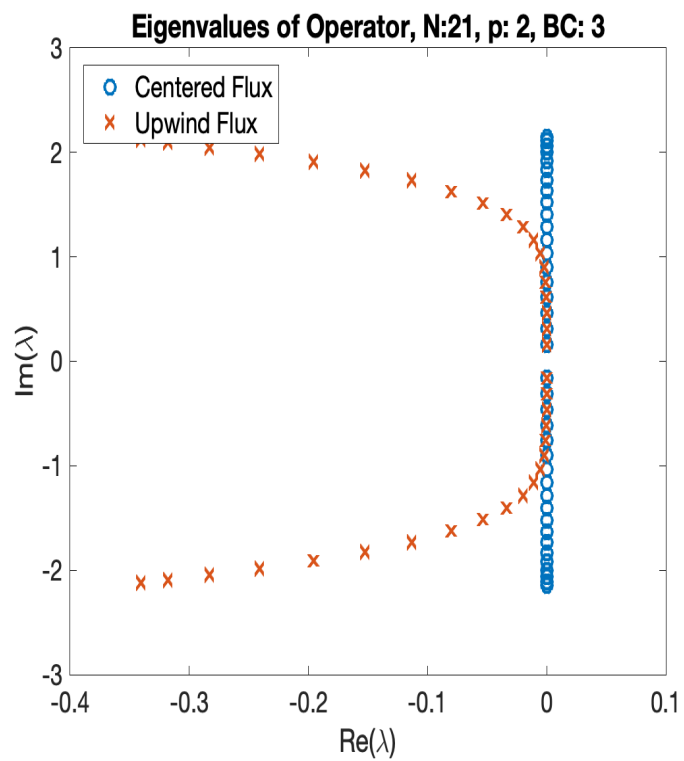
- As we use them simply as a basis for DG (or CG) methods all standard theory (energy stability, approximation) applies. Combination with standard methods on structured-unstructured grids is automatically stable.
- As finite difference methods they are compact schemes - the mass matrix is the tensor product of banded matrices so time derivatives can be computed in linear time.
- Time step stability constraints (at high order) are an order of magnitude better than for finite element methods of comparable order.



In the interior the methods superconverge at roughly twice the design order, $2p$ rather than $p + 1$, with (for the upwind penalty in an interior penalty method) dissipation of order $2p + 1$. We can see this by looking at the numerical dispersion relation. For example with $p = 4$ and $\eta = kh$:

$$ik_h = ik \left(1 + \frac{93937}{464486400} \eta^8 + \mathcal{O}(\eta^{10}) \quad + \quad \frac{9i}{32768} \eta^9 + \mathcal{O}(\eta^{11}) \right)$$

Also, the largest eigenvalues (with extrapolation boundary closures) grow slowly with p :



How can we extend all of this to numerical relativity? Using the Landau-Lifschitz formulation we derive a system of the form:

$$\left(\frac{\partial}{\partial t} + \sum_j W^j \frac{\partial}{\partial x_j} \right)^2 h^{\mu\nu} - \sum_{j,k} P^{jk} \frac{\partial^2 h^{\mu\nu}}{\partial x_j \partial x_k} = \bar{S}^{\mu\nu} + \bar{F}^{\mu\nu}.$$

Here:

$$W^j = \frac{h^{j4}}{2(1 + h^{44})}, \quad P^{jk} = (1 + h^{44})^{-1} \left(\delta^{jk} - h^{jk} + \frac{h^{j4}h^{k4}}{4(1 + h^{44})} \right).$$

and

$$\begin{aligned} \sqrt{-g}g^{\mu\nu} &= \eta^{\mu\nu} - h^{\mu\nu}, \\ \frac{\partial h^{\mu 4}}{\partial t} + \sum_j \frac{\partial h^{\mu j}}{\partial x_j} &= 0. \end{aligned}$$

Dervng the energy-DG form is direct. The issue is the choice of upwind flux taking into account the “advection” term W . For W small enough we can prove stability but we need a more sophisticated choice in general - work in progress.

THANK YOU!